The Monocentric City Model

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Graduate Urban Economics, Lecture 2
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Student presentations: Lecture 1

1. Papers on Zipf's Law in China, including: Luckstead and Devadoss (Ec. Letters 2014), Soo (Papers in Regional Science 2014), or others (get my approval first)
2. Card, Rothstein, Yi, "Location, Location, Location," *U.S. Census Bureau Working Paper*, 2021
3. Combes, Demurger, Li, Wang, "Unequal Migration and Urbanisation Gains in China," *Journal of Development Economics*, 2020
4. Combes, Demurger, Li, "Migration Externalities in Chinese cities," *European Economic Review*, 2015
5. Dingel, Miscio, Davis, "Cities, Lights, and Skills in Developing Economies," *Journal of Urban Economics*, 2020

Student presentations: Lecture 2

1. Harari, Mariaflavia, “Cities in Bad Shape: Urban Geometry in India,” *American Economic Review*, 2020
2. Zheng, Siqi and Kahn, Matthew, “Land and residential property markets in a booming economy: New evidence from Beijing,” *Journal of Urban Economics*, 2008
3. Zhou, Zhengyi, Chen, Hong, Han, Lu, and Zhang, Anming, “The Effect of a Subway on House Prices: Evidence from Shanghai,” *Real Estate Economics*, 2021
4. Gupta, Arpit, Van Nieuwerburgh, Stijn, and Kontokosta, Constantine, “Take the Q Train: Value Capture of Public Infrastructure Projects,” *Journal of Urban Economics*, 2022
5. Liu, Crocker, Rosenthal, Stuart, and Strange, William, “The Vertical City: Rent Gradients, Spatial Structure, and Agglomeration Economies,” *Journal of Urban Economics*, 2018

Student presentations: Lecture 2, cont...

1. Rosenthal, Stuart, Strange, William, and Urrego, Joaquin, “Are City Centers Losing Their Appeal? Commercial Real Estate, Urban Spatial Structure, and COVID-19,” *Journal of Urban Economics: Insight*, 2022
2. Delventhal, Matthew, Kwon, Eunjee, and Parkhomenko, Andrii, “How do cities change when we work from home?” *Journal of Urban Economics: Insight*, 2022

Introduction

One of the biggest questions in urban economics–and urban studies generally–is what forces drive the spatial distribution of a city's population?

1. Are there any general distribution patterns that seem to hold across cities and countries
2. What is the relationship between housing prices, land rents, and density?
3. What determines which types of people (ex: income, job type) live where?
4. Can we predict the effect of policy changes?

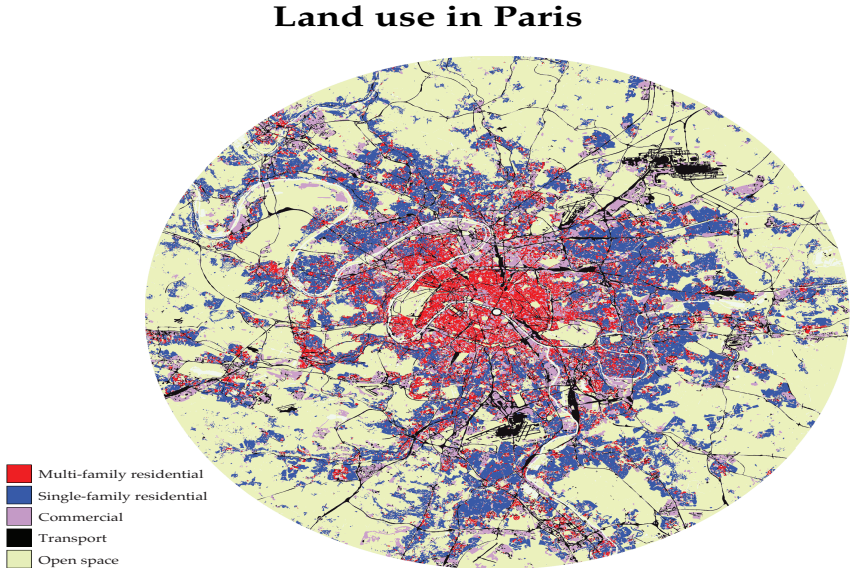
To get a better idea of these questions let's look at some maps

Source: Gilles Duranton November 7, 2015 presentation and Alain Bertaud, Feb 2002 presentation

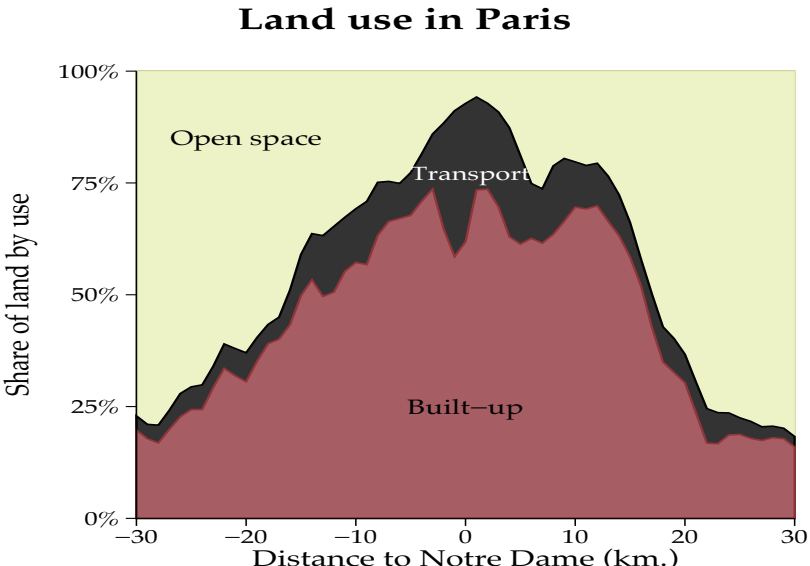
Land Price Map, Source: Duranton

Source: Ahlfeldt, Redding, Sturm, and Wolf (2014)

Land Use Maps, Source: Duranton



Land Use Maps, Source: Duranton



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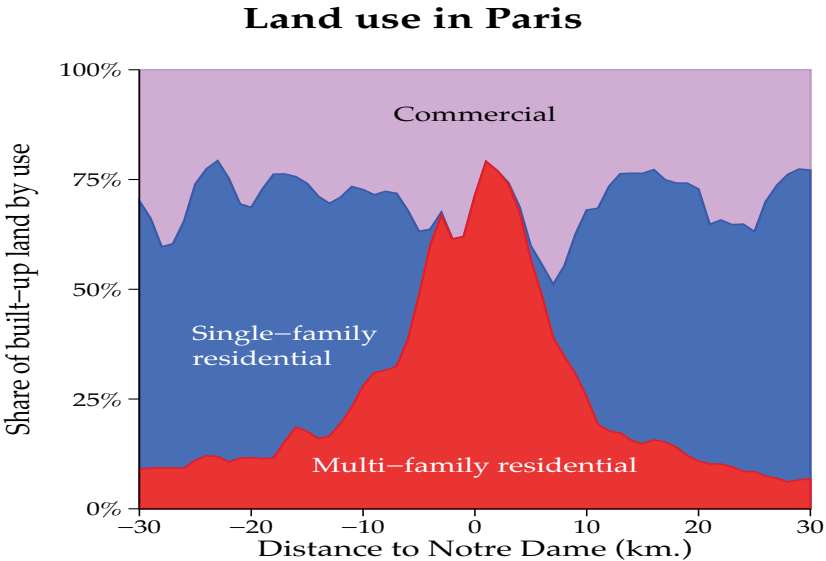
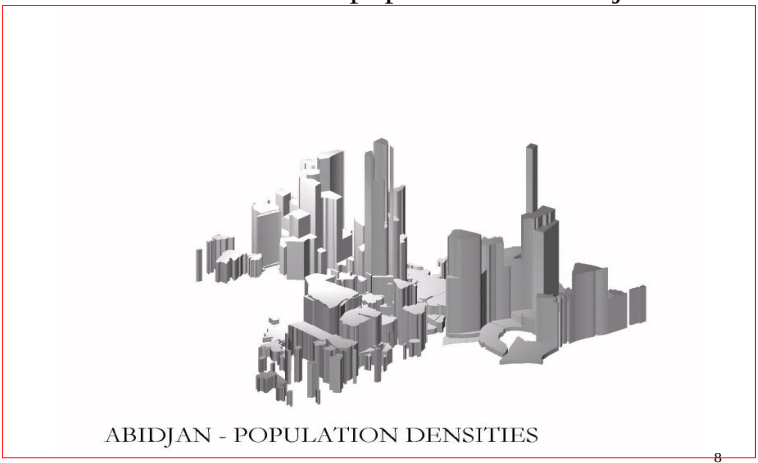


3D Density Map, Source: Bertaud

Distribution of population in Abidjan



3D Density Map, Source: Bertaud

Distribution of population in Hong Kong

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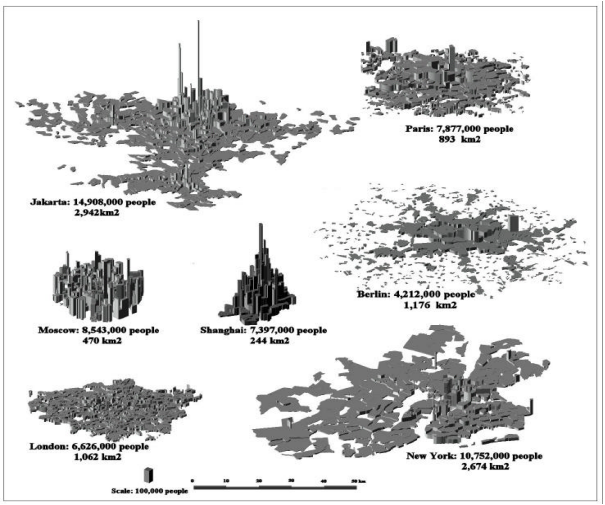
3D Density Map, Source: Bertaud

Spatial distribution of population in 7 major metropolis represented at the same scale



Common Patterns

There are very different densities across these cities–do you see any common patterns?

One thing we generally see is a pattern of declining density radiating from one center, or sometimes multiple centers

From the Berlin map we saw a similar pattern with land prices

What are the main forces that could generate this pattern? What happens when these change?

Specific questions we could ask:

1. What should happen to the spatial distribution of population as Shanghai builds more subway lines extending into far districts?
2. What should happen to Shanghai residents' quality of life as transportation infrastructure improves? Does it depend on the hukou system?

Monocentric City Model: goals and main idea

Very famous Urban Economics model, developed by William Alonso (1964), Edwin Mills (1967), and Richard Muth (1969)

The goal of the model is to explain the spatial distribution of population in a city.

Main mechanism is the relationship between commuting costs, housing price, and housing consumption

We are interested in deriving a set of *gradients* and comparative statics

Results:

1. Housing prices decrease with distance from the Central Business District (CBD)
2. Housing consumption increases with distance from CBD
3. Density and capital-to-land ratio decrease with distance from CBD

Model Framework

- All residents are identical, consume housing and numeraire good
- Price of numeraire good does not vary with location but housing price can
- All jobs are located in center of city (CBD), residents commute with positive transportation cost
- Land is owned by absentee landlords (not in city), developers rent land and build housing in perfectly competitive market with CRS production
- In equilibrium: all residents have same utility, everyone must have housing in city

In my slides I will mostly use Brueckner's notation with matching equation labels.

Residents Maximization Problem

- Consumers have utility $v(z, q)$ over numeraire z and housing q
- Commuting cost is $\tau * x$, where x is distance from CBD
- Given wage y and housing price $p(x)$, budget constraint: $z + p(x) * q(x) + \tau * x = y$
- All residents have equilibrium utility u

Resident utility maximization problem is thus:

$$\max_{q} v(y - \tau * x - p(x)q(x), q) = u \tag{1}$$

Intuition from Undergrad Version

Fix housing consumption to $q(x) = \bar{q}$ for all residents, now:

$$\max_x v(y - \tau * x - p(x)\bar{q}, \bar{q}) = u \tag{1a}$$

FOC gives:

$\frac{\partial v}{\partial z}\left(-\tau - \frac{\partial p}{\partial x}\bar{q}\right) = 0$ or $\frac{\partial p}{\partial x} = \frac{-\tau}{\bar{q}}$

The price *per unit* of housing must decline with distance to compensate for increased commuting so that all residents have same consumption of z

Ex: if $p(0)$ is price at center then it must be that $y - p(0) * \bar{q} = y - \tau * x - p(x) * \bar{q}$, which is true when $p(x) = p(0) - \frac{\tau}{\bar{q}} * x$

Undergrad Bid-Rent

Since housing is fixed the equal utility condition implies all residents consume same $z = \bar{z}$

We can ignore utility and just assume residents will bid against each other for different locations x; what is max price $p(x)$ any resident would pay?

$\bar{z} = y - \tau * x - p(x) * \bar{q}$, or $p(x) = \frac{y - \tau * x - \bar{z}}{\bar{q}}$

This gives the same gradient $\frac{\partial p}{\partial x} = \frac{-\tau}{\bar{q}}$

This is called the "bid-rent" approach, useful later

We can close basic undergrad model by assuming a fixed population N so that boundary of city is $\bar{x} = N * \bar{q}$ where price is $\bar{p}$

Residents' Optimal Consumption

$$\max_{q} v(y - \tau * x - p(x)q(x), q) = u \tag{1}$$

This maximization problem leads to two conditions: 1) optimization 2) equal utility

Optimization implies the MRS is equal to ratio of prices:

$$\frac{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial q}}{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial z}} = \frac{p}{1} \tag{2}$$

The equal utility condition implies:

$$v(y - \tau * x - p(x)q(x), q) = u \tag{3}$$

Deriving the Price Gradient

$$\frac{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial q}}{\frac{\partial v(y-\tau*x-p(x)q(x),q)}{\partial z}} = \frac{p(x)}{1} \tag{2}$$

$$v(y - \tau * x - p(x)q(x), q(x)) = u \tag{3}$$

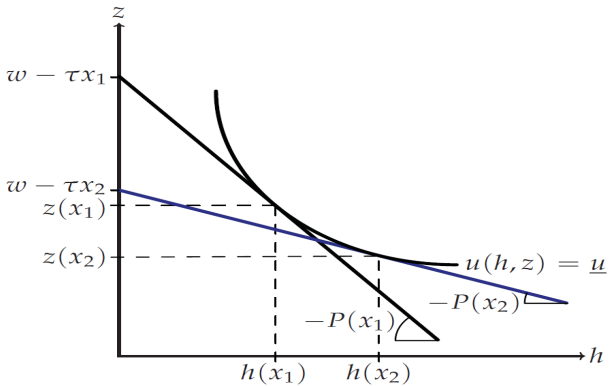
If we totally differentiate eq. 3 wrt x:

$$\frac{\partial v}{\partial z} * \left(-\tau - \frac{\partial p(x)}{\partial x} q(x) - p(x) \frac{\partial q(x)}{\partial x} \right) + \frac{\partial v}{\partial q} * \frac{\partial q(x)}{\partial x} = 0 \tag{4}$$

Plugging eq 2 into 4 (last term cancels with last term in parentheses–envelope theorem) yields:

$$\frac{\partial p(x)}{\partial x} = \frac{-\tau}{q(x)} \tag{5}$$

Intuition for Price Gradient: Duranton+Puga 2015



Panel (b)
Comparative statics

Consumers must be optimizing and have same utility at all x

Price Gradient: Alonso-Muth Condition

$$\frac{\partial p(x)}{\partial x} = \frac{-\tau}{q(x)} \tag{5}$$

Price declines with distance from the center as a function of transportation costs and housing

If we forced all residents to consume equal amounts of housing $q(x) = \bar{q}$ then the *gradient* (slope wrt distance) is constant: prices must decrease linearly so that all consumers have equal income (since they have equal consumption)

If housing increases with dist from CBD then gradient is convex: consumers substitute cheaper housing consumption for numeraire consumption, so prices don't have to decline as quickly to compensate consumers

The Bid-Rent Function

Another way to solve the model is to reframe the consumer's problem in term of "bid-rent": the maximum price $p(x)$ consumers are willing to pay for housing at location x such that utility is u

Useful approach because when agents are heterogeneous (next class) to figure out who lives where: agents with the highest bid-rent function for a location outbid other agents

Bid-rent:

$$\Psi(x,u) \equiv \max_{q(x),z(x)} p(x) | v(q,z) = u, y - \tau * x = p(x)q(x) + z(x)$$

Plugging in budget constraint gives:

$$\Psi(x,u) = \max_{q(x),z(x)} \left\{ \frac{y - \tau * x - z(x)}{q(x)} | v(q,z) = u \right\}$$

Alternative Derivation of Alonso-Muth

$\Psi(x,u) = \max_{q(x),z(x)} \left\{ \frac{y-\tau*x-z(x)}{q(x)} | v(q,z) = u \right\}$

A key condition is that utility is always equal to u, therefore we can rewrite above equation using Hicksian demand functions from expenditure minimization problem:

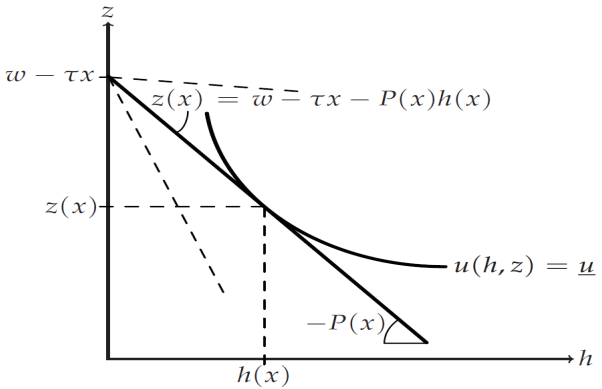
$$\min_{q,z} z + p(x) * q(x), \text{s.t. } v(z,q) = u \tag{1b}$$

Now: $\Psi(x,u) = \max_{q(x)} \left\{ \frac{y-\tau*x-z(q(x),u)}{q(x)} \right\}$

Then, taking derivative wrt x and using the envelope theorem:

$\frac{dp(x)}{dx} = \frac{d\Psi(x,u)}{dx} = -\frac{\tau}{q(x)}$

Intuition for Bid-Rent: Duranton+Puga 2015



Panel (a)
Deriving housing prices in x

Note: budget constraint pivots around z-intercept (not a shift, as when deriving expenditure function)

Housing Consumption Gradient

In this model housing price $p(x)$ adjusts so that all residents have equal utility

Therefore we can work with either Marshallian housing demand $q(p(x), y)$ or Hicksian demand $q(p(x), u)$

Gradient of Hicksian housing demand is:

$$\frac{dq(p,u,x)}{dx} = \frac{\partial q}{\partial p} * \frac{\partial p}{\partial x} > 0$$

Therefore, we know that housing consumption is *increasing* with distance; the housing price is cheaper so consumers substitute towards housing

Housing Production

The housing construction industry is perfectly competitive with a *concave* CRS production function

Input to construction is land L and capital K: $H(K, L)$

The important part of concavity is that $H_{KK} < 0$; building higher is more expensive

The price of capital is i, price of land at x is $r(x)$

It will turn out to be easier to work with the capital-to-land ratio: $S = K/L$

Then, because we assume CRS we can write $\frac{H(K,L)}{L} = H(K/L, L/L) = H(S, 1)$

Define $h(S) \equiv H(S, 1)$ as housing-per-unit-land

Profit: $\Pi(x) = L * (p(x) * h(S) - i * S - r(x))$

Firm optimization and market structure

With CRS and free entry we have a perfectly competitive market with construction firms earning zero profit

Similar to the utility maximization problem, this gives two conditions: 1) FOC for optimal S and 2) zero-profit equation

$$p(x) * \frac{\partial h(S)}{\partial S} = i \tag{11}$$

$$p(x) * h(S) - i * S(x) - r(x) = 0 \tag{12}$$

Totally differentiating these conditions will allow us to derive the land-rent gradient and capital-to-land ratio gradient

Deriving land rent and capital-to-land gradient

$$p(x) * \frac{\partial h(S)}{\partial S} = i \tag{11}$$

$$p(x) * h(S) - i * S(x) - r(x) = 0 \tag{12}$$

Define ϕ as the set of parameters $\phi = x, \tau, y, u$

Totally differentiating gives:

$$\frac{\partial p}{\partial \phi} * \frac{\partial h}{\partial S} + p * \frac{\partial^2 h}{(\partial S)^2} * \frac{\partial S}{\partial \phi} = 0 \tag{13}$$

$$(p * \frac{\partial h}{\partial S} - i) * \frac{\partial S}{\partial \phi} + \frac{\partial p}{\partial \phi} h = \frac{\partial r}{\partial \phi} \tag{14}$$

Land rent and capital-land gradient

Finally, by inserting the FOC (11) into (14) we get:

$$\frac{\partial r}{\partial \phi} = h * \frac{\partial p}{\partial \phi} \tag{15}$$

Re-arranging (13) gives:

$$\frac{\partial S}{\partial \phi} = -\frac{\partial h}{\partial S} * (p * \frac{\partial^2 h}{(\partial S)^2})^{-1} * \frac{\partial p}{\partial \phi} \tag{16}$$

Our earlier concavity assumption implies that $\frac{\partial^2 h}{(\partial S)^2} < 0$

This gives us:

$$\frac{\partial r}{\partial x} < 0, \text{ and }, \frac{\partial S}{\partial x} < 0 \tag{17}$$

Population Density

Assume every person lives in a separate house

Then the population at x is the total amount of housing at x divided by the per-person consumption of housing: $N(x) = H(x)/q(x)$

The population density (pop/land) is thus: $D(x) = H(x)/(L * q(x)) = h(s)/q(x)$

Differentiating: $\frac{\partial D(x)}{\partial x} = \frac{\partial h(S)}{\partial S} * \frac{\partial S(x)}{\partial x} * \frac{1}{q(x)} - \frac{h(S)}{q(x)^2} * \frac{dq}{dx} < 0$

Density decreases for *two* reasons: 1) capital-to-land ratio declines 2) per-person housing consumption increases

Summary of Results

$$\frac{\partial p}{\partial x} = \frac{-\tau}{q(x)} < 0 \tag{r1}$$

$$\frac{dq}{dx} = \frac{\partial q(p,u)}{\partial p} * \frac{\partial p}{\partial x} > 0 \tag{r2}$$

$$\frac{\partial r}{\partial x} = h(S) * \frac{\partial p}{\partial x} < 0 \tag{r3}$$

$$\frac{\partial S}{\partial x} = -\frac{\partial h}{\partial S} * (p * \frac{\partial^2 h}{(\partial S)^2})^{-1} * \frac{\partial p}{\partial x} < 0 \tag{r4}$$

$$\frac{\partial D(x)}{\partial x} = \frac{\partial h(S)}{\partial S} * \frac{\partial S(x)}{\partial x} * \frac{1}{q(x)} - \frac{h(S)}{q(x)^2} * \frac{dq}{dx} < 0 \tag{r5}$$

Comparative Statics

What happens to spatial distribution as income, transportation cost, population, agricultural land rent, or utility change?

Comparative statics of the model depend upon our assumption about migration

Closed City Model: no migration, population is exogenous and equilibrium utility u is determined by the model

Open City Model: free migration no moving frictions, implies "spatial equilibrium condition" that utility must be equal in every city; population is endogenous and adjusts to ensure (exogenous) equal utility u

I'll discuss how to close the model and the intuition, see proofs in Brueckner

Equilibrium Conditions

To cut down on algebra and still maintain intuition we assume: 1) All land can be developed $L(x) = 1$, and 2) City is on a line instead of area of circle (1 dimension instead of 2)

Two equilibrium conditions we use to close model:

1) Residents out-bid farmers for use of land, which means city ends at some $\bar{x}$ where land rent is equal to agricultural land rent

$$r(\bar{x}, y, \tau, u) = r_A \tag{18}$$

2) Everyone (population N) is housed within boundary of city ($\bar{x}$)

$$\int_0^{\bar{x}} D(x)dx = N \tag{19}$$

Note: equation 19 is simpler than in Brueckner due to above assumptions

Land Rent at CBD

Following Duranton and Puga (2015) we use:

d1) $\frac{\partial p(x)}{\partial x} = \frac{-\tau}{q(x)}$ d2) $\frac{\partial r(x)}{\partial x} = h(S) * \frac{\partial p}{\partial x}$

Then density is: $D(x) = -\frac{1}{\tau} * \frac{\partial r}{\partial x}$

This is a very useful way to write density because then:

$$\int_0^{\bar{x}} D(x)dx = \int_0^{\bar{x}} -\frac{1}{\tau} * \frac{\partial r(x)}{\partial x}dx = \frac{r(\bar{x}) - r(0)}{-\tau} = N \quad (1)$$

Land rent differential—rent at CBD vs fringe—is thus proportional to population and transportation cost: $r(0) - r_A = \tau * N$

Note: above only holds for linear city

Solving Closed City Model

From the developer's problem we can write price as a function of land rent (eq 12):

$p(x) = \frac{iS(r)+r}{h(S(r))} = C(i, r)$

Zero-profit condition means unit price of housing equals unit cost of housing $C(i, r)$, then:

$p(0) = C(i, r(0)) = C(i, r_A + \tau * N)$

Then, since utility is equal at all locations, $v(x) = v(p(0), y) = u$

With u we can then solve for $p(x)$ function, last task is to find $\bar{x}$

We know price at fringe must be equal to construction cost at fringe, can invert to find $\bar{x}$:

$p(\bar{x}) = C(i, r(\bar{x})) = C(i, r_A)$

Closed City: Increase in Agricultural Rent

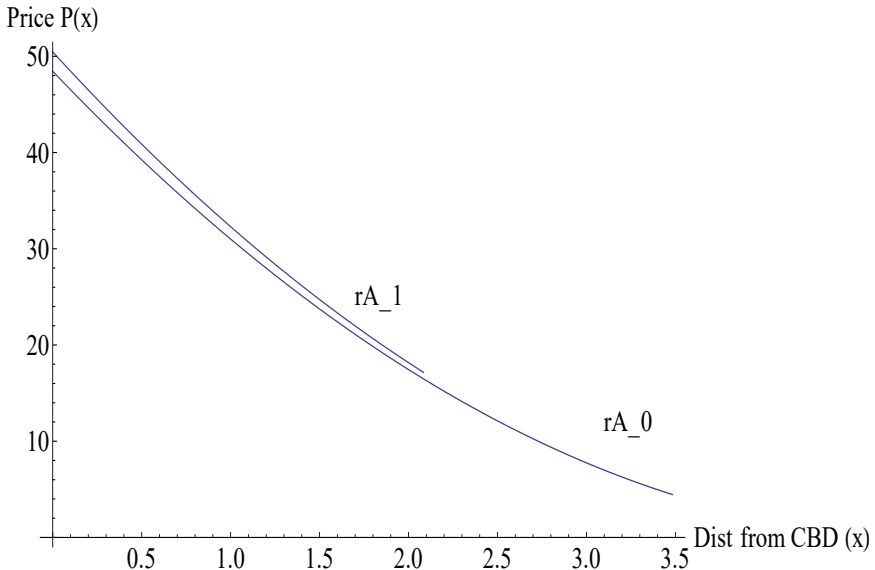
What happens to u, $\bar{x}$, price and density gradients?

1. Equilibrium utility decreases
2. Fringe contracts $\bar{x}_1 < \bar{x}_0$
3. Price gradient *shifts up* and steeper
4. Density rises everywhere

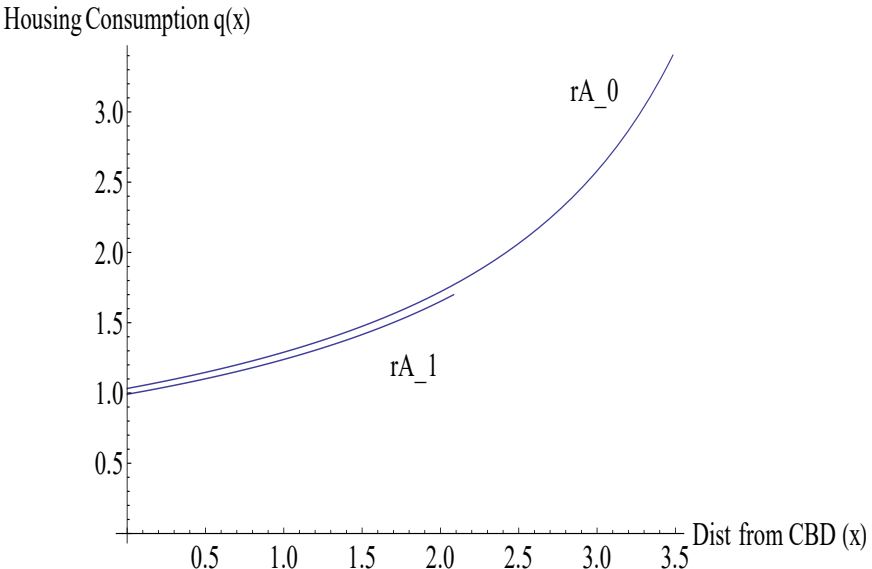
Basically cuts city at new r_A, everyone must live in smaller area

Q: Why can't the fringe $\bar{x}$ stay the same?

Example: Closed City, Agricultural Rent Increase



Example: Closed City, Agricultural Rent Increase



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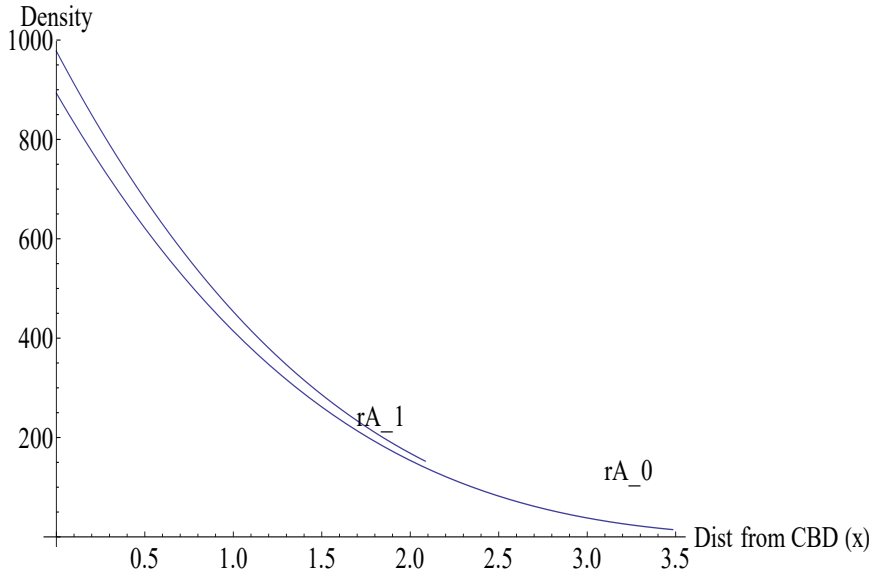


Closed City: Population Increase

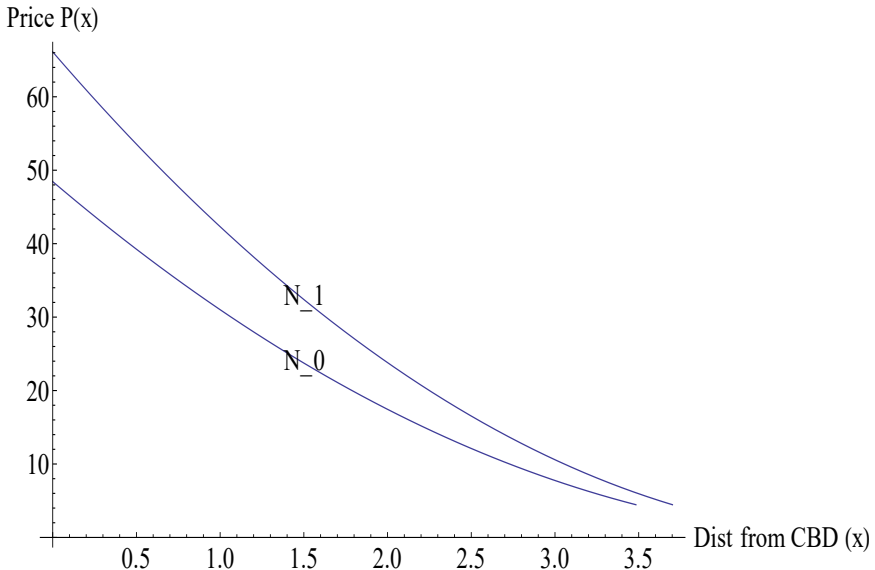
What happens to u, $\bar{x}$, price and density gradients?

1. Equilibrium utility decreases
2. Fringe expands $\bar{x}_1 > \bar{x}_0$
3. Price gradient *shifts up* and steeper
4. Density rises everywhere–why not just for $x > \bar{x}_0$?

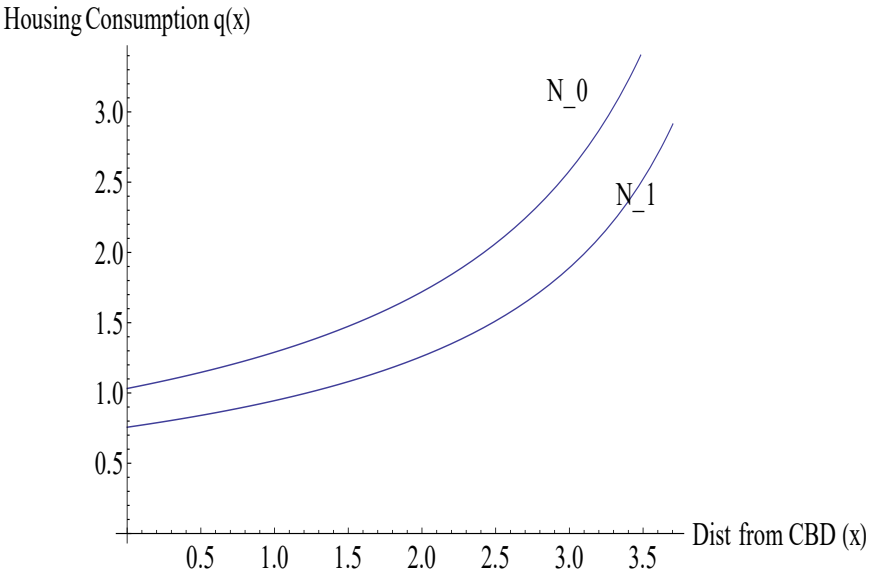
City expands geographically but not enough so that density at x is constant

Intuition (example): new residents locating outside of fringe would have lower utility than at fringe, thus bid-up prices. Increase in price makes more central locations desirable, all locations will increase. Generally, more room to increase price closer to center because housing consumption is smaller, but also depends on functional form assumptions (ex: how convex is cost of building higher)

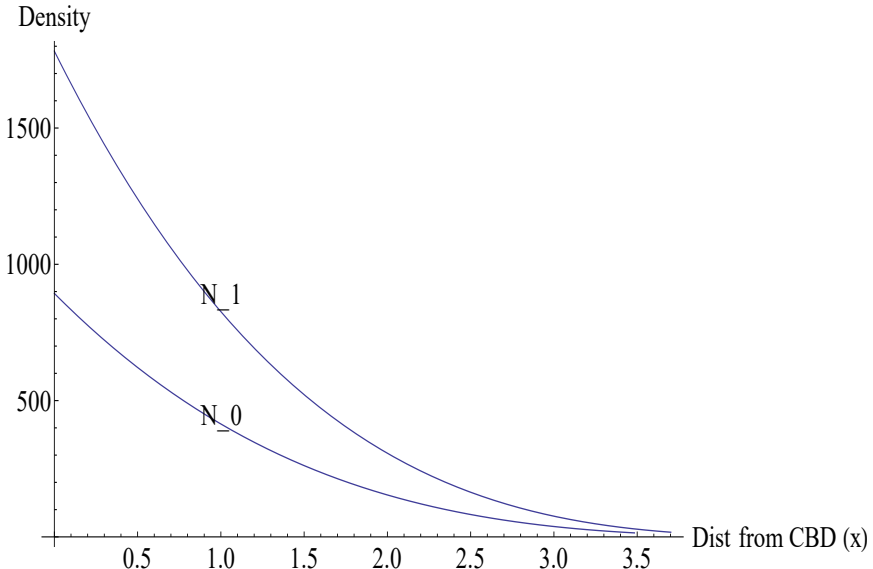
Example: Closed City, Population Increase



Example: Closed City, Population Increase



Example: Closed City, Population Increase



Closed City: Income Increase

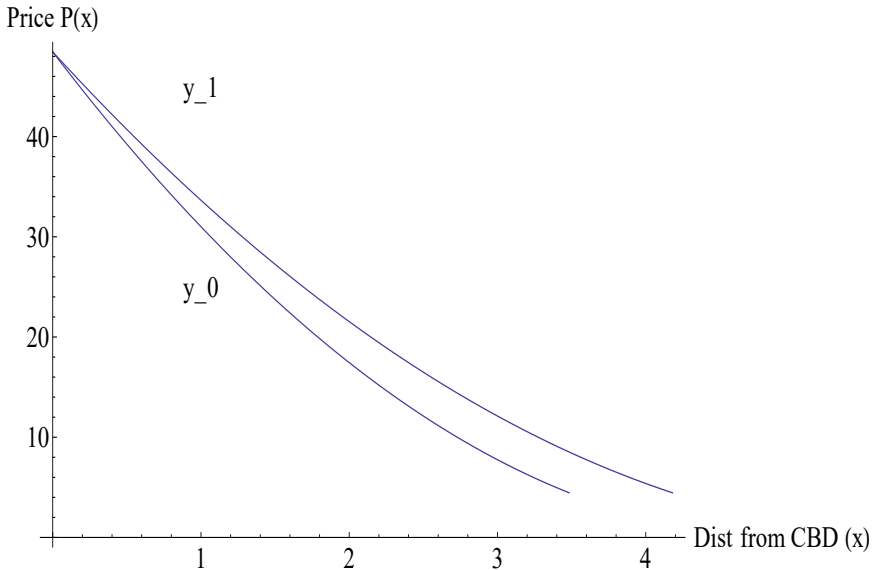
Income and transportation cost changes are complicated because there are both indirect effects through utility (same as pop and fringe rent), *but also* direct effects:

$$\frac{dp}{dy} = \frac{\partial p}{\partial u} * \frac{\partial u}{\partial y} + \frac{\partial p}{\partial y}$$

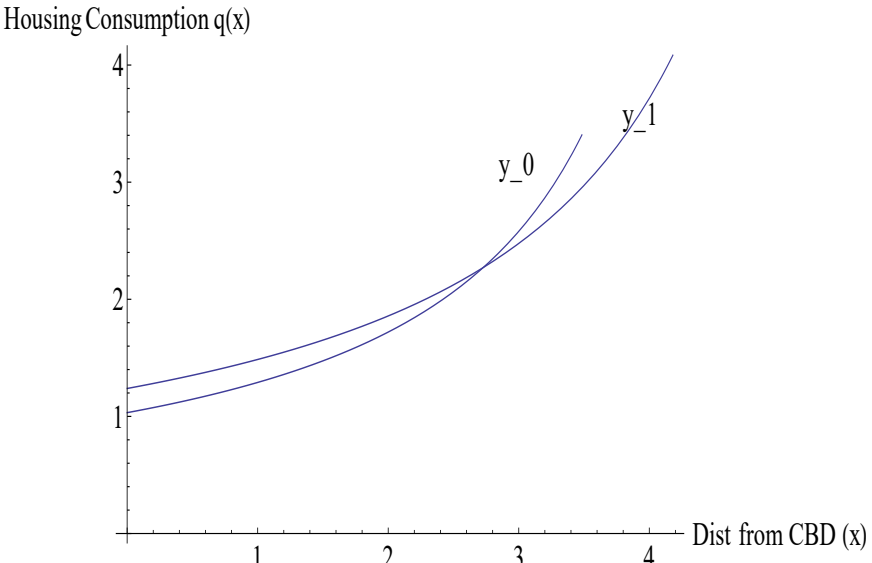
1. Equilibrium utility increases
2. Fringe expands $\bar{x}_1 > \bar{x}_0$
3. Price gradient *rotates*; because we assumed linear city it rotates at center. In 2d city can rotate away from center (as in Brueckner article)
4. Density gradient *rotates*; note that it drops at center just enough so that price is same despite increase in housing consumption (same amount of housing, fewer people)
5. *For this functional form* housing consumption gradient also rotates; tradeoff between housing and numeraire consumption (technical detail)

City expands geographically, most people consume more housing, live further away, increases density away from CBD

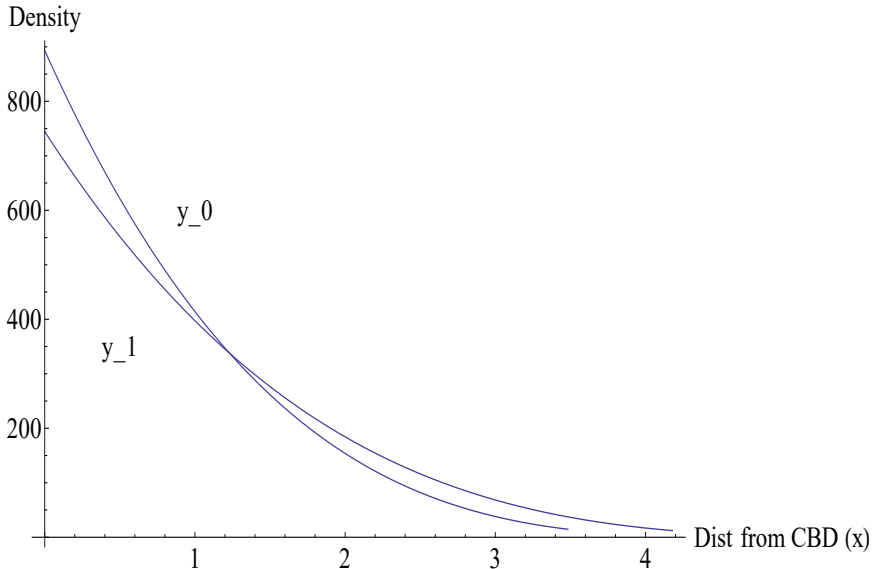
Closed City, Income Increase, Price Gradient



Closed City, Income Increase, Housing Gradient



Closed City, Income Increase, Density Gradient



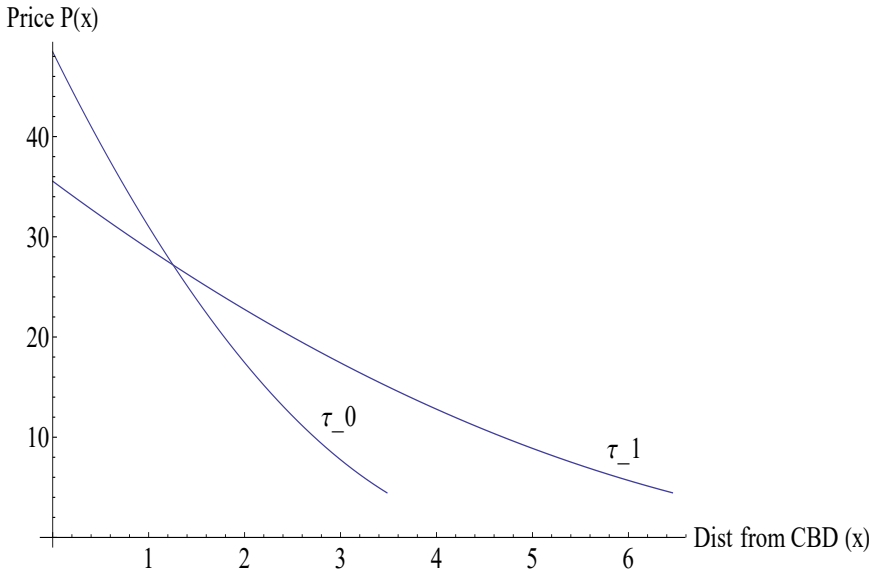
Closed City: Decrease in Transportation Cost

What happens to u, $\bar{x}$, price and density gradients?

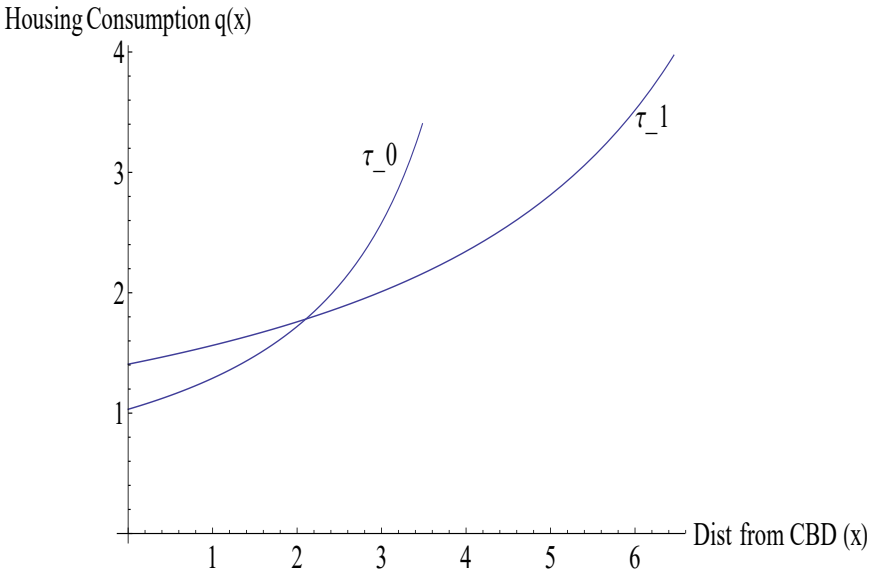
1. Equilibrium utility increases
2. Fringe expands $\bar{x}_1 > \bar{x}_0$
3. Price gradient *rotates*: for $x < x^*$ $p(x)$ declines, $x > x^*$ $p(x)$ increases
4. Where price falls density falls, density rises where price rises

Basically more distant locations become more attractive, decreasing demand for central locations

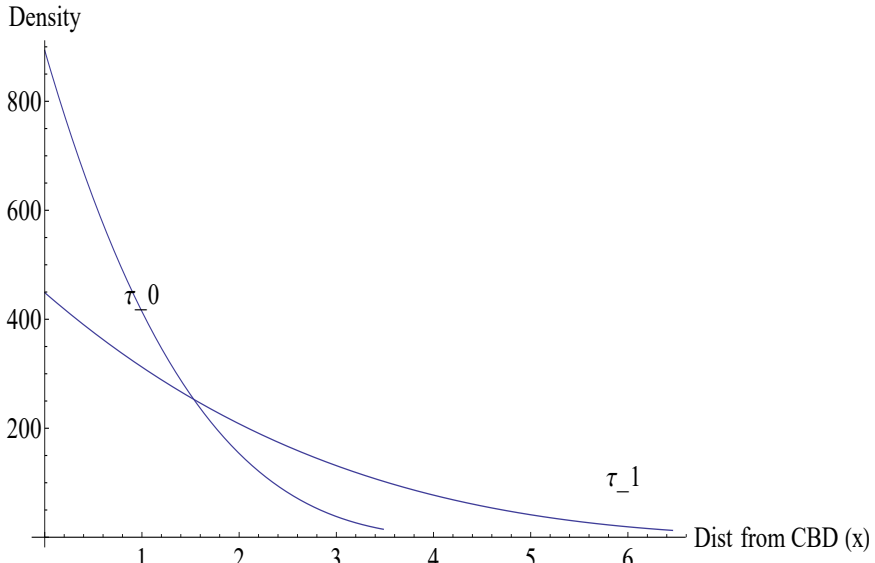
Example: Closed City, Transportation Cost Decrease



Example: Closed City, Transportation Cost Decrease



Example: Closed City, Transportation Cost Decrease



Open City Comparative Statics

Open city comparative statics are easier because utility is fixed due to population flow

We can therefore compute comparative statics directly from the variable equations, without worrying about the indirect effect of utility

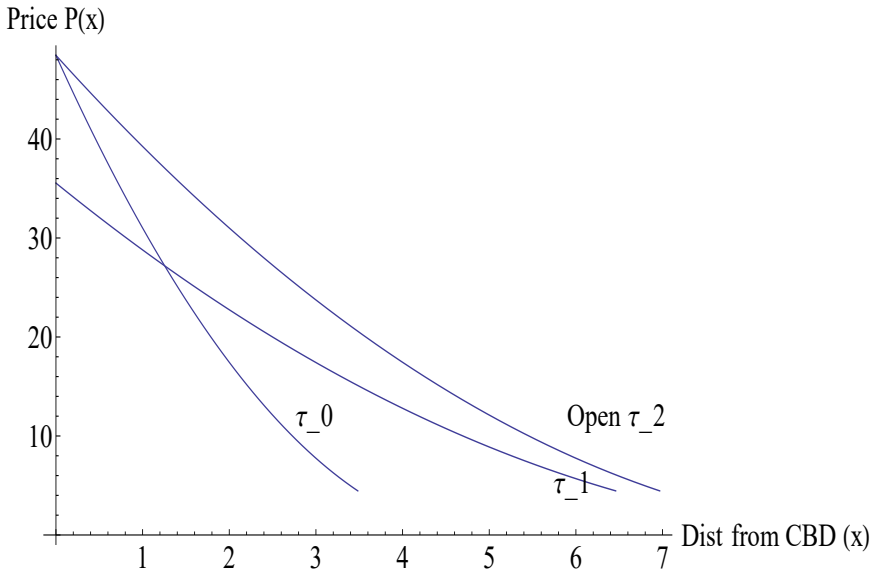
For example: decrease in transportation cost must raise prices (to ensure utility doesn't change), expand the fringe, decrease housing consumption, and thus lead to higher density and a higher overall population

Can also interpret open city comparative statics as "short-run" and "long-run":

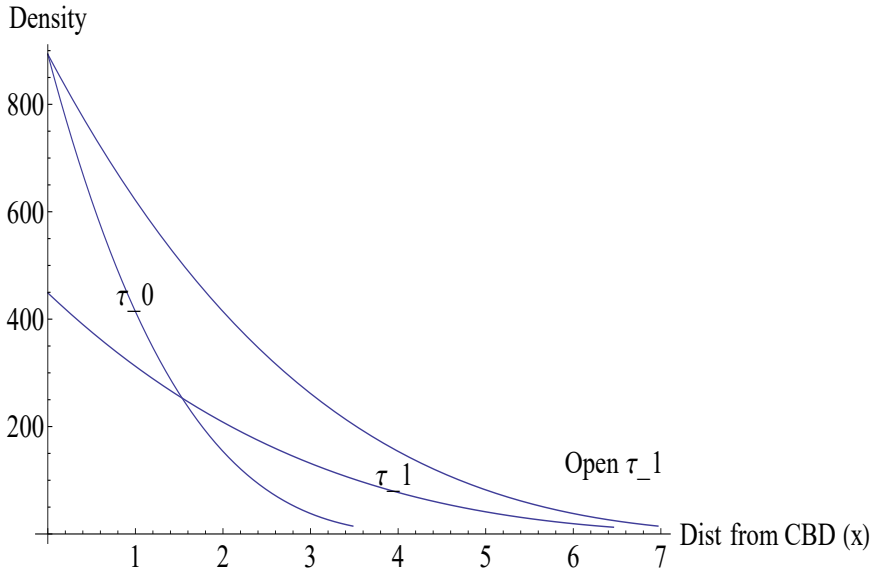
First, the parameter change induces the closed city equilibrium, which has a different utility

Then, population flows in or out to restore original utility level, with resulting closed-city effect of population change (long-run)

Ex: OPEN City, Transportation Cost Decrease



Ex: OPEN City, Transportation Cost Decrease



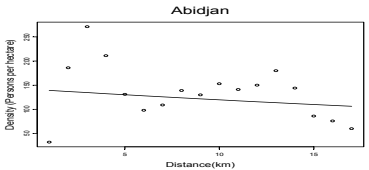
Some empirical gradients

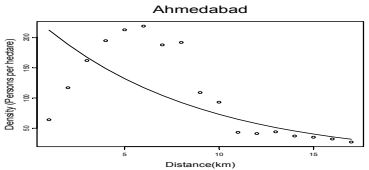
The following are density estimates from various cities collected by Bertaud and Malpezzi.

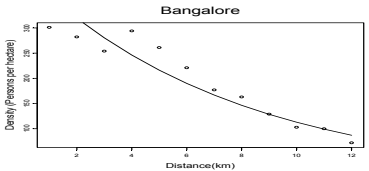
The data is not public, CBD definition is subjective,year is unclear; still quite informative

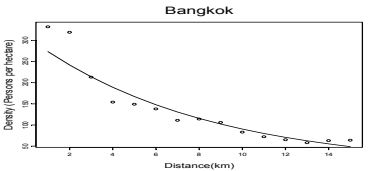
Source: Bertaud and Malpezzi, “The Spatial Distribution of Population in 48 World Cities: Implications for Economies in Transition”, World Bank Report 2003

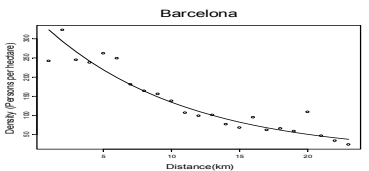
Empirical Density Gradients

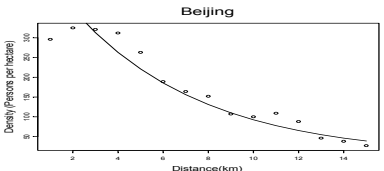




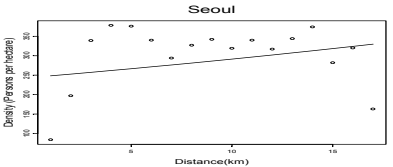


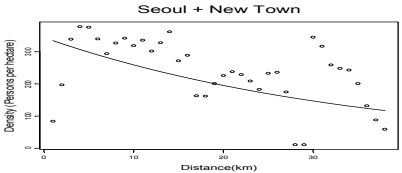


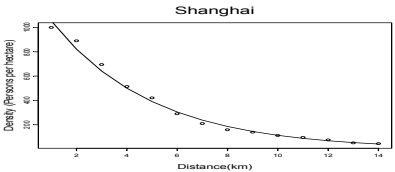


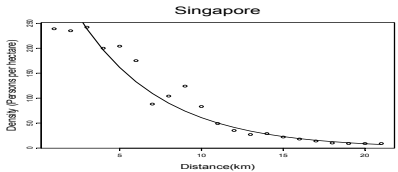


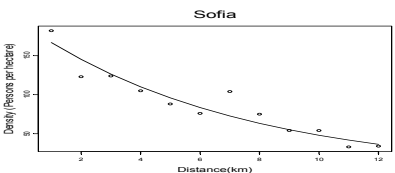
Empirical Density Gradients

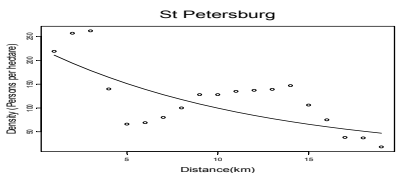




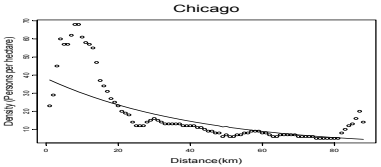


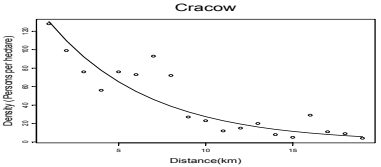


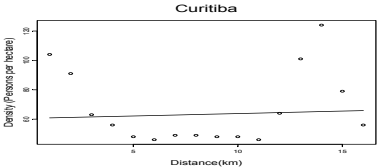


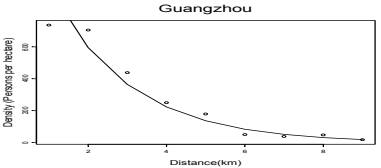


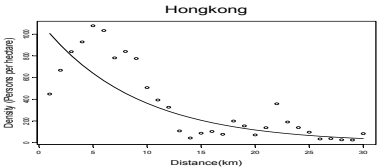
Empirical Density Gradients

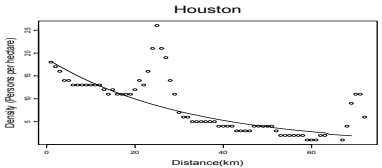












Concluding: Weaknesses of the Model

Elegant analytical framework has a cost: some unrealistic and non-trivial assumptions

1. Everyone commutes to job in CBD; many cities are polycentric (multiple job centers)
2. Housing stock is perfectly flexible; in fact, housing stock is quite durable and this durability is important
3. No zoning or regulations; empirical work argues these frictions can be significant
4. Residents are identical (this can be relaxed somewhat–different bid rent curves by type)

Nonetheless, a very important and flexible model, continues to be widely used

Next Class: Read Jerch et. al. article; LeRoy and Sonstelie may help with theoretical intuition (both articles on my website)

References for this Lecture

This lecture is based on the following references:

1. Brueckner,Jan K., *Handbook of Regional and Urban Economics*, Volume 2, Ch. 20, 1987
2. Fujita, Masahisa, *Urban Economic Theory*, Ch. 2-3, 1989
3. Duranton, Gilles and Puga, Diego, *Handbook of Economic Growth*, Volume 2B, Ch. 5, 2014
4. Duranton, Gilles and Puga, Diego, *Handbook of Regional and Urban Economics*, Volume 5, 2015
5. Duranton, Gilles, *Empirics of housing, land use, and location choice*, Presentation given at Frontiers of Urban Economic Conference, November 7, 2015